Induced matter: Curved *N*-manifolds encapsulated in Riemann-flat *N*+1 dimensional space

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Liko and Wesson have recently introduced a new five-dimensional induced matter solution of the Einstein equations, a negative curvature Robertson-Walker space embedded in a Riemann-flat five-dimensional manifold. We show that this solution is a special case of a more general theorem prescribing the structure of certain $N+1$ dimensional Riemann-flat spaces which are all solutions of the Einstein equations. These solutions encapsulate N-dimensional curved manifolds. Such spaces are said to "induce matter" in the submanifolds by virtue of their geometric structure alone. We prove that the N-manifold can be any maximally symmetric space. [DOI: 10.1063/1.2042968]

The concept of "induced matter," was originally introduced by Wesson.[1,2] While investigating five-dimensional (5D) Kaluza-Klein theory, he recognized that a curved 4-space could be embedded in a Ricci-flat ($R_{AB}=0$; $A,B,\ldots \in \{0,1,2,3,4\}$) 5-space. This is a reflection of the Campbell-Magaard theorem[3] which, applied to 5D, states that it is always possible to embed a curved four-dimensional (4D) manifold in a 5D Ricci-flat space. Seahra and Wesson[4] provide an overview and rigorous proof of the Campbell-Magaard theorem with applications to higher dimensions. Wesson takes "induced matter" to mean that the left-hand geometric side extra terms of the flat 5D Ricci-tensor provide the source terms in the 4D curved Ricci-tensor of the embedded space. A "weak" version of this concept utilizing an embedding of the Friedmann-Roberston-Walker (FRW) 4-space in a Minkowski 5-space has been used to visualize the big bang sectionally.[5] Here, the 5-space is Riemann-flat ($R_{ABCD}=0$) since it is Minkowski. There is no physics in the 4D subspace, except with reference to the original FRW coordinates. This simply provides a Euclidean embedding diagram.

More recently, Liko and Wesson have introduced a new 5D, Riemann-flat solution[6] which they found could "encapsulate" a 4D curved FRW space. We use the term "encapsulate" as distinct from embed since in this 5-space, the coordinates are not Minkowski. The 4D subspace is itself curved in the same 5D coordinates. It is true that a flattening transformation can be found to a 5D Minkowski space. However, this would simply produce another embedding diagram. The physics seems to lie in the encapsulating 5D metric. We shall use the term "induced matter" to include a Riemann-flat 5D manifold encapsulating a curved 4D subspace. The Liko-Wesson induced matter solution goes on to describe an apparently inflationary universe as a negative curvature FRW space embedded in a special 5D universe. The RW space undergoes accelerated expansion subject to a repulsive "dark energy" ($P=-\rho$). We will show in this paper that the Liko-Wesson solution is a special case of a more general class of maximally symmetric submanifolds embedded in Riemann-

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flat space. A detailed discussion of maximally symmetric submanifolds based on Poincarè metrics and their consequences can be found in Ref. 7. For convenience, we repeat some critical definitions and calculations.

Consider the Riemann manifold defined by

$$\mathrm{d}S^2 = \tilde{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{1}$$

This space is said to be maximally symmetric if and only if it has constant sectional curvature $\kappa=\kappa(i,j)$, for any $1\leqslant i\neq j\leqslant N$. In the plane spanned by the basis vectors $(\hat{e}_i,\hat{e}_j)$ the sectional curvature is defined by

$$\kappa(i,j) = \tilde{g}^{ii}R^j_{iji} \quad (i,j \text{ not summed}). \tag{2}$$

For a maximally symmetric space $R^j_{iji}=\kappa\tilde{g}_{ii}$, $j\neq i$. For such a space,

$$R_{ii} = -\kappa(N-1)\tilde{g}_{ii}. \tag{3}$$

Theorem: Let $\tilde{g}_{ij}$ represent a maximally symmetric space of sectional curvature κ. *The metric*

$$\mathrm{d}S^2 = \mathrm{d}\tau^2 - D\tau^2\tilde{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \quad i,j\ldots, \in \{1,2,\ldots,N\}, \tag{4}$$

is Riemann-flat whenever $D=-\kappa$.

Proof: Consider the metric

$$\mathrm{d}S^2 = \mathrm{d}\tau^2 - f(\tau)^2\tilde{g}_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{5}$$

where $\tilde{g}_{ij}$ denotes a maximally symmetric space. We compute the independent components of the curvature tensor (the overtilde denotes differentiation in τ):

$$R^i_{0j0} = -\frac{f''}{f}\delta^i_j,$$

$$R^0_{i0j} = -ff''\tilde{g}_{ij},$$

$$R^k_{ikj} = \tilde{R}_{ij} - (N-1)f'^2\tilde{g}_{ij} = -\kappa(N-1)\tilde{g}_{ij} - (N-1)f'^2\tilde{g}_{ij} = -(N-1)(f'^2+\kappa)\tilde{g}_{ij}, \tag{6}$$

where we have made use of the result (3). It is evident from (6) that the space will be Riemann-flat if and only if $f''=0$ and $f'^2+\kappa=0$. Let $f(\tau)=\sqrt{D}\tau$. Then $f''=0$ and $f'^2-D=0$. It follows that $D=-\kappa$ and the proof is complete.

Liko and Wesson[6] introduce the line element (with overall sign of $\mathrm{d}S^2$ reversed from ours),

$$\mathrm{d}S^2 = \mathrm{d}\tau^2 - \frac{\tau^2}{L^2}\left[\mathrm{d}t^2 - L^2\sinh^2\!\left(\frac{t}{L}\right)\mathrm{d}\sigma_3^2\right], \tag{7}$$

where

$$\mathrm{d}\sigma_3^2 = \left(1+\frac{kr^2}{4}\right)^{-2}(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2), \tag{8}$$

is the Robertson-Walker 3-space with $k=-1$.

Define coordinates $x^A=\{\tau,r,\theta,\phi,t\}$, $A\in\{0,1,2,3,4\}$.

We can then identify in (4),

$$\tilde{g}_{ii} = \{-f, -fr^2, -fr^2 \sin^2 \theta, 1\}, \quad f = L^2 \sinh^2(t/L)\left(1 + \frac{kr^2}{4}\right)^{-2}, \tag{9}$$

and $D=1/L^2$. Equation (7) is thus of the form (4) and will satisfy the theorem provided that the sectional curvature of the 4-space is $\kappa=-1/L^2$. Direct evaluation of the sectional curvature for two typical cases (by symmetry, the remaining cases are identical) results in

$$\kappa(1,2) = \tilde{g}^{11} R^2_{121} = -\frac{1}{L^2},$$

$$\kappa(4,2) = \tilde{g}^{44} R^2_{424} = -\frac{1}{L^2}$$

which show that the conditions (6) are met. That is, the 4-space has constant sectional curvature which then results in a Riemann-flat 5-space. We have thus shown that the new metric solution, (7), introduced by Liko and Wesson is a special case of our more general theorem (4) which allows the N-space to be any maximally symmetric manifold.

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