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An electrodynamic connection

Harry I Ringermacher
United Technologies Research Center, East Hartford, CT 06108, USA

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Abstract. Starting from a first principles basis deformation of a Riemannian manifold, a new affine connection is derived that contains both gravitational and electromagnetic fields on equal footing term for term. Using this connection, it is shown that parallel transport of the tangent vector on a curve yields the Lorentz force equation for a charged particle, while transport of the normal vector yields the equation of motion for a classical spin including the covariant Thomas precession. Furthermore, the new electrodynamic connection exactly satisfies the form and the compatibility criteria for the general non-symmetric connection sought by Schrödinger for his affine field theory. The results of the present work are shown to be equivalent to introducing an electrodynamic torsion on the manifold thus rendering it non-Riemannian. The new connection may serve to illuminate the geometry of electrodynamics.

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1. Introduction

We shall develop in this paper a new non-symmetric affine connection, incorporating gravitational and electromagnetic forces, grounded in the theory of the deformation of tensor manifolds of Dienes [1]. The electromagnetic 4-vector potential acts as the deformation vector in the new geometry. The new connection will be shown to be exactly the general non-symmetric connection that Schrödinger prescribed as the solution form for his original affine theory and satisfies his 'compatibility' condition [2]. Furthermore, it will be shown that it can be independently derived from the assumption of a particular electrodynamic torsion present in the space according to the geometric development of Schouten [3].

Although this work was initially motivated by a mathematical parallel between electromagnetism and a Riemannian generalization of the classical theory of elasticity of Green and Zerna [4] and Coburn [5], it became clear that point deformations, the foundation of elasticity theory, were inadequate to describe the known physics. This is particularly evident in the failure of Weyl's theory from lack of length invariance. A more general formalism can be found in the theory of deformation of tensor manifolds of Dienes who treats both point and basis deformations. We shall use basis deformation theory, not as an end in itself, but as a means to illuminate a new route to a possible embedding of the electromagnetic fields into a metric geometry. The end result of this paper will be the development of a new non-symmetric affine connection which has several remarkable properties.

We begin, in section 2, by developing Dienes' basis deformation theory and then showing how it can be extended to yield a 'physical' affine connection. In sections 3 and 4, we shall apply the new connection, via a vector transport process, to generate the Minkowski equation of motion for a massive point charge as well as the equation of motion

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for a classical spin including the Thomas precession. The latter result is new—from a geometric standpoint. In section 5 we show that the connection appears to match what Schrödinger sought for his early affine theory. In section 6 the connection is derived from a torsion introduced on the manifold.

The new 'electrodynamic connection' is the central geometric object developed in this paper. Its significance in the context of Schrödinger's work is evident and may further illuminate it, but the present work is also an example of the physical significance of torsion, a subject of recent interest [6, 7].

2. Deformation theory

As mentioned earlier, Dienes develops both the theory of point deformations and the theory of basis deformations. He defines two states of a Riemannian manifold—initial undeformed and final deformed. His point deformation theory, which distinguishes the initial undeformed from the final deformed coordinates, leads to the generalized theory of elasticity on a Riemannian manifold. When developing his basis deformation theory, however, Dienes does not distinguish between the undeformed and the deformed coordinates since only a basis deformation is involved at a given point. We shall show that when this distinction is retained, physically meaningful results follow.

We shall adopt the basis deformation theory of Dienes. The standards used generally follow Adler [8] and Barut [9].

We begin by providing some preliminary definitions that clearly describe the initial undeformed system. Following Dienes, we require that orthonormality of the basis and parallel transport be preserved under the deformation. The latter condition means that a parallel vector field in the undeformed state remains parallel in the deformed state. This will permit us to correctly define the deformed metric tensor and to obtain a simple deformed connection with respect to this metric. Length is not preserved in the deformed system, hence the remaining task will be to develop a new 'effective deformed' connection that can represent the deformation referenced to the initial undeformed state.

The connection of the metric tensor $g_{\mu\nu}$ of signature (-2) is given by

$$\Gamma^{\lambda}_{\mu\nu} = -\tfrac{1}{2} g^{\alpha\lambda}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\nu\mu}). \tag{1}$$

We introduce the tetrad e_μ. Each of the $e_\mu (\mu = 0, 1, 2, 3)$ is a 4-vector. Absolute differentiation of a covariant basis e_μ is given by

$$\mathrm{D}e_\mu = \mathrm{d}e_\mu + \Gamma^{\lambda}_{\mu\nu}\,\mathrm{d}x^\nu e_\lambda. \tag{2}$$

The metric tensor is defined from

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{3}$$

where

$$g_{\mu\nu} = e_\mu \cdot e_\nu = \eta_{ij} e^i_\mu e^j_\nu \qquad \text{and} \qquad e^\mu \cdot e_\nu = \delta^\mu_\nu \qquad e^\mu_i e_{j\mu} = \eta_{ij}.$$

Summation convention is used throughout and η_{ij} is the Minkowski tensor. In passing to the deformed system we may equivalently deform either the space via the metric tensor or the coordinate differentials. If both are deformed simultaneously, we shall find that, under the covariant and contravariant deformation transforms defined below, both the length of a vector and the interval remain invariant. However, we shall not proceed with simultaneous deformations since this will also get us nowhere. We choose instead to first deform the metric allowing us to define a deformed connection under preservation of parallel transport. Having defined such a connection in the procedure described below, we are in a position to carry out inverse deformations on the vectors and coordinates back to the initial coordinates in such a way as to extract a meaningful connection that describes the deformed state from the undeformed coordinates. This is a fairly complicated procedure where all coordinates are tracked under both basis deformations and parallel transport of vectors from an initial deformed state to a final undeformed state and a new 'effectively deformed' connection is obtained. For this reason, the full calculation is relegated to the appendix and only the definitions and relevant results are shown below. The deformed system is defined from:

$$G_{\mu\nu} = \boldsymbol{E}_\mu \cdot \boldsymbol{E}_\nu \qquad \boldsymbol{E}^\mu \cdot \boldsymbol{E}_\nu = \delta^\mu_\nu \qquad E^\mu_i E_{j\mu} = \eta_{ij}. \tag{4}$$

Following Dienes, we introduce the covariant and contravariant basis deformations:

$$\boldsymbol{E}_\lambda = \boldsymbol{e}_\lambda - \kappa \boldsymbol{k}_\lambda \qquad \boldsymbol{E}^\lambda = \boldsymbol{e}^\lambda + \kappa \boldsymbol{k}^\lambda. \tag{5}$$

These relations consisting of the gravitational basis $\boldsymbol{e}_\mu$ combined with what later is shown to be an electromagnetic deformation $\boldsymbol{k}_\mu$ are fundamental and follow strictly from the basis orthogonality requirement under the deformation. The quantity $\kappa \boldsymbol{k}_\mu$ is assumed to be small where κ is a constant to be determined. The present theory assumes as infinitesimal deformation so that only terms of order κ are retained.

It follows that $\boldsymbol{k}_\nu$ and $\boldsymbol{k}^\alpha$ are related by the raising and lowering operations:

$$\boldsymbol{k}_\nu = \boldsymbol{e}_\alpha \boldsymbol{e}_\nu \cdot \boldsymbol{k}^\alpha \qquad \boldsymbol{k}^\nu = \boldsymbol{e}^\alpha \boldsymbol{e}^\nu \cdot \boldsymbol{k}_\alpha. \tag{6}$$

The lowering operation, for example, can also be written in component notation:

$$k_{i\nu} = e_{i\alpha} e_{j\nu} k^{j\alpha}. \tag{7}$$

The Latin indices refer to the scalar components of each 4-vector. It is evident from this that the relations (6) can be written as deformation matrix operations upon the $\boldsymbol{e}$ basis:

$$\boldsymbol{k}_\nu = \zeta^\alpha_\nu \boldsymbol{e}_\alpha \qquad \boldsymbol{e}_\nu \cdot \boldsymbol{k}^\alpha \equiv \zeta^\alpha_\nu. \tag{8}$$

It can be shown that in the present case

$$\boldsymbol{e}_\nu \cdot \boldsymbol{k}^\alpha = \boldsymbol{e}^\alpha \cdot \boldsymbol{k}_\nu \tag{9}$$

so that there is no ambiguity in the matrix definition. Thus from relations (6) and (8) we may rewrite the relations (5) as deformation transforms for covariant and contravariant vectors. For covariant vectors, applied to our tetrad, we have:

$$\boldsymbol{E}_\lambda = D^\alpha_\lambda \boldsymbol{e}_\alpha \qquad D^\alpha_\lambda \equiv \delta^\alpha_\lambda - \kappa \zeta^\alpha_\lambda. \tag{10}$$

For contravariant vectors:

$$\boldsymbol{E}^\alpha = \bar{D}^\alpha_\lambda \boldsymbol{e}^\lambda \qquad \bar{D}^\alpha_\lambda \equiv \delta^\alpha_\lambda + \kappa \zeta^\alpha_\lambda. \tag{11}$$

δ is the Kronecker symbol. The inverse deformations are obtained simply by changing κ to $-\kappa$. These operations will be used extensively in the appendix. From relations (4) and (5) we readily find the deformed metric tensor to order κ:

$$G_{\mu\nu} = g_{\mu\nu} - \kappa(k_\mu \cdot e_\nu + e_\mu \cdot k_\nu) \qquad G^{\mu\nu} = g^{\mu\nu} + \kappa(k^\nu \cdot e^\mu + e^\nu \cdot k^\mu). \tag{12}$$

From these, a connection may be defined with respect to the deformed metric by substituting G from the relations (12) for g in expression (1). All terms to order κ are retained leaving:

$$\bar{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \tfrac{1}{2}\kappa[e^\lambda \cdot (k_{\nu;\mu} + k_{\mu;\nu}) + e_\nu \cdot (k_{\sigma;\mu} - k_{\mu;\sigma})g^{\lambda\sigma} + e_\mu \cdot (k_{\sigma;\nu} - k_{\nu;\sigma})g^{\lambda\sigma}]. \tag{13}$$

The covariant derivatives are with respect to Γ, the undeformed connection.

It can be shown, however, that this connection does not yield the correct equation of motion for a charged particle under parallel transport. The presence of the antisymmetric k factors in the second and third terms in the brackets—shortly to be related to the electromagnetic field tensor—is spoiled by the symmetric factor in the first term. It is this term that is responsible for the problems. However, Γ is the connection relative to the deformed metric and it is physically essential to reference the undeformed state where lengths are preserved. We therefore must find how this connection will appear with respect to the undeformed state by using our new basis transport properties. The full calculation is carried out in the appendix.

The final 'effective' deformed connection is;

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \tfrac{1}{2}\kappa[e^\lambda \cdot (k_{\nu;\mu} - k_{\mu;\nu}) + e_\nu \cdot (k_{\sigma;\mu} - k_{\mu;\sigma})g^{\lambda\sigma} + e_\mu \cdot (k_{\sigma;\nu} - k_{\nu;\sigma})g^{\lambda\sigma}]. \tag{14}$$

This connection is the foundation of all of our remaining work. Our evaluation of the deformed connection, equation (13), with respect to the undeformed state has introduced exactly the right correction term (appendix A) to make the symmetry of the first term in the brackets consistent with the others. The immediate consequence, however, is the non-symmetric connection, equation (14).

Our task will now be to ascribe some physical meaning to this new connection. This is most transparent if we parallel transport the tangent vector under the effective deformation. We shall call this operation a 'parallel deformation' and it will enable us to identify the deformation vector k_ν with electrodynamic quantities.

3. Equation of motion for a point charge

We are now in a position to identify k_ν and the constant κ. Absolute differentiation of any vector ξ^λ with respect to the new connection is given by

$$\tilde{\mathrm{D}}\xi^\lambda = \mathrm{d}\xi^\lambda - \tilde{\Gamma}^\lambda_{\mu\nu}\,\mathrm{d}x^\nu\,\xi^\mu. \tag{15}$$

We shall define a parallel deformation from the vanishing of the absolute derivative just as in parallel transport so that ξ^λ describes a 'geodesic';

$$\mathrm{d}\xi^\lambda = \tilde{\Gamma}^\lambda_{\mu\nu}\,\mathrm{d}x^\nu\,\xi^\mu. \tag{16}$$

This immediately yields the equation of motion of the tangent vector, u^μ, with respect to the undeformed metric and coordinates;

$$\frac{\mathrm{d}u^\lambda}{\mathrm{d}s} = \tilde{\Gamma}^\lambda_{\mu\nu} u^\nu u^\mu. \tag{17}$$

Inserting the deformation connection (14) and rearranging terms to define the absolute derivative with respect to the undeformed connection on the left-hand side yields the general result;

$$\frac{\mathrm{D}u^\lambda}{\mathrm{d}s} = \tfrac{1}{2}\kappa u^\mu u^\nu [e^\lambda \cdot (k_{\nu;\mu} - k_{\mu;\nu}) + e_\nu \cdot (k_{\sigma;\mu} - k_{\mu;\sigma}) g^{\lambda\sigma} + e_\mu \cdot (k_{\sigma;\nu} - k_{\nu;\sigma}) g^{\lambda\sigma}]. \quad (18)$$

For $\kappa = 0$, this equation is the usual geodesic of general relativity. Otherwise the right-hand side represents 'forces' acting along the path.

Let us now expand the invariant 4-velocity and 4-vector potential in the basis e_μ:

$$\boldsymbol{u} = u^\mu \boldsymbol{e}_\mu \qquad \boldsymbol{A} = A^\mu \boldsymbol{e}_\mu. \quad (19)$$

We shall choose as our covariant and, according to the raising operation equation (6), our contravariant deformation vectors, the direct products of the velocity and vector potential:

$$\boldsymbol{k}_\nu = \boldsymbol{u} A_\nu \qquad \boldsymbol{k}^\nu = u^\nu \boldsymbol{A}. \quad (20)$$

This choice represents the single ansatz of our theory. Inserting this into equation (17) yields

$$\frac{\mathrm{D}u^\lambda}{\mathrm{d}s} = \tfrac{1}{2}\kappa u^\mu u^\nu [e^\lambda \cdot \boldsymbol{u}(A_{\nu;\mu} - A_{\mu;\nu}) + e_\nu \cdot \boldsymbol{u}(A_{\sigma;\mu} - A_{\mu;\sigma}) g^{\lambda\sigma} + e_\mu \cdot \boldsymbol{u}(A_{\sigma;\nu} - A_{\nu;\sigma}) g^{\lambda\sigma}] \quad (21)$$

or

$$\frac{\mathrm{D}u^\lambda}{\mathrm{d}s} = \tfrac{1}{2}\kappa u^\mu u^\nu [u^\lambda (A_{\nu;\mu} - A_{\mu;\nu}) + u_\nu (A_{\sigma;\mu} - A_{\mu;\sigma}) g^{\lambda\sigma} + u_\mu (A_{\sigma;\nu} - A_{\nu;\sigma}) g^{\lambda\sigma}]. \quad (22)$$

In arriving at this result we have passed $\boldsymbol{u}$ through the covariant differentiation with respect to the undeformed connection. We can do this since the covariant derivatives of the basis vectors vanish and the u^ν are vector fields along a curve. Upon carrying out the summations on ν and μ, the first term on the right vanishes by symmetry while the second and third terms are identical upon interchanging dummy indices leaving

$$\frac{\mathrm{D}u^\lambda}{\mathrm{d}s} = -\kappa u^\mu F^\lambda_\mu. \quad (23)$$

This is the Minkowski equation for a charge if the electromagnetic field tensor is identified as

$$F_{\lambda\mu} = A_{\mu;\lambda} - A_{\lambda;\mu} \quad (24)$$

and $\kappa = -e/mc^2$, where e and m are the test particle charge and mass. Thus, the test particle as well as the external field is intimately coupled to the local deformation and the resulting geometry.

Using the deformation vectors of equation (18) together with equation (14) defines the complete effective deformation connection;

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \tfrac{1}{2}\kappa g^{\lambda\sigma}(u_\mu F_{\nu\sigma} + u_\nu F_{\mu\sigma} - u_\sigma F_{\nu\mu}). \quad (25)$$

A complete expansion reveals the remarkable symmetry between the electromagnetic and gravitational components:

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = -\tfrac{1}{2}g^{\alpha\lambda}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} + \partial_\alpha g_{\nu\mu}) + \tfrac{1}{2}\kappa g^{\alpha\lambda}(u_\mu F_{\nu\alpha} + u_\nu F_{\mu\alpha} - u_\alpha F_{\nu\mu}). \tag{26}$$

We write this more compactly

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \kappa\Lambda^{\lambda}_{\mu\nu} \tag{27}$$

where

$$\Lambda^{\lambda}_{\mu\nu} = \tfrac{1}{2}g^{\alpha\lambda}(u_\mu F_{\nu\alpha} + u_\nu F_{\mu\alpha} - u_\alpha F_{\nu\mu}).$$

This connection is the central new result of this paper. It is the basis not only for describing the usual motion of the structureless test charge above but also, as will be shortly demonstrated, for describing the correct behaviour of a classical spin. It is interesting to observe in hindsight that this connection could have been constructed by inspection from the Lorentz force equation alone with the exception, perhaps, of the antisymmetric third term which does not contribute to the charge motion. This term is nevertheless critical for our work as will be seen in the next section.

Another point is worth noting here. Physical electromagnetic force terms appear alongside gravitational force terms in the connection. This is in sharp contrast to Weyl's theory, for example, where the vector potentials appear in the connection. Thus a metric theory based upon this connection should introduce electromagnetic potentials alongside gravitational potentials in the metric solutions. Failure in this regard has in fact been a puzzling point in past theories. Recently it has been experimentally verified that the gravitational *potential* introduces quantum phase shifts in the neutron wavefunction [10] in interferometry experiments and so it would seem that electromagnetic potentials, known to produce phase shifts in electron wavefunctions via the Aharonov–Bohm effect, should be on equal footing with the gravitational potentials and not with the gravitational forces as they are in all Weyl-type theories.

As a check into the physical meaning of this connection, we evaluate the $\{i, 0, 0\}$ component,

$$\tilde{\Gamma}^{i}_{00} = \partial^i \Phi_g + \frac{e}{m}E^i \tag{28}$$

which is seen to be the sum of the gravitational and electric accelerations of the test particle and is the natural generalization of the usual gravitational field intensity term. In the next section we shall develop the first new dynamics resulting from this geometric structure as applied to a 'classical' spin.

4. Equation of motion for a classical spin

In the previous section we parallel deformed the tangent vector of the space and were able to identify the consequent equation of motion exactly with the Lorentz force law for a massive point charge thus determining κ, the single parameter of this theory. It is reasonable then to ask if there are other vectors that, upon undergoing a parallel deformation, result in physically meaningful motion under the same parameter. One possibility is, of course, the acceleration vector $\dot{u}^\mu$ which is the principal normal to the path. This may very well be meaningful in the context of radiative effects but is beyond the scope of the present work. There is permitted, however, a complete orthonormal tetrad according to the Frenet–Serret equations for a curve [11]. In addition, these can all be related to Fermi–Walker transport which is discussed in detail by Synge [12] and Misner [13] so that clearly there is room for other possibilities.

Let us therefore represent this vector S^μ in the most general way by simply requiring that it be normal to the tangent vector;

$$S^\mu u_\mu = 0. \tag{29}$$

Applying a parallel deformation to S^μ according to equations (16) and (27) gives:

$$\frac{\mathrm{d}S^\mu}{\mathrm{d}s} = \Gamma^\mu_{\alpha\beta} u^\beta S^\alpha + \tfrac{1}{2}\kappa (S_\sigma u^\sigma) u^\beta F_\beta^{\cdot\mu} + \tfrac{1}{2}\kappa F_\alpha^{\cdot\mu} S^\alpha - \tfrac{1}{2}\kappa u^\mu u^\beta F_{\beta\alpha} S^\alpha. \tag{30}$$

Using the orthogonality condition, equation (29), generates the exact equation of motion for a 'classical spin', S^α, with a g-factor of one:

$$\frac{\mathrm{d}S^\mu}{\mathrm{d}s} = \Gamma^\mu_{\alpha\beta} u^\beta S^\alpha + \tfrac{1}{2}\kappa F_\alpha^{\cdot\mu} S^\alpha - \tfrac{1}{2}\kappa u^\mu u^\beta F_{\beta\alpha} S^\alpha. \tag{31}$$

The first term is, of course, the gyroscopic precession in a gravitational field and can be combined with the left hand side to represent an absolute derivative. The second term is the usual gyromagnetic precession while the third term is precisely the covariant Thomas precession. Most importantly, the Thomas term follows directly from the third antisymmetric term of our Λ connection, equation (27), which clearly demonstrates the generality of this expression. An extensive discussion of 'classical spin' can be found in Barut [9], p 73 and Jackson [14]. The spin equation is normally derived by seeking a covariant generalization of the rest frame gyromagnetic equation as was done by Bargmann *et al* [15] or can be developed from a Lagrangian in the manner of Barut. The point is that this equation appears almost trivially from a parallel deformation. Geometrically speaking, the terms look like a gyromagnetic rotation vector together with a pure Fermi transport vector—the Thomas precession.

What we have done so far is to create a new connection from strictly geometric first principles and show that this connection relates to the two well known equations of motion for charge and magnetic moment. However, we have not as yet demonstrated any relationship between this work and any prior work that might place the present results on a more firm foundation. There is, in fact, just such a link. In the next section we shall show that the electrodynamic connection precisely matches the prescription for the general non-symmetric connection sought by Schrödinger in his later 'unified' affine theory and, as such, serves to further illuminate it.

5. Schrödinger compatibility and the electrodynamic connection

In the late 1940s, Schrödinger concentrated on several attempts to achieve a unified field theory without success as he, himself, insisted. These efforts are described by Hittmair in a Schrödinger centenary volume edited by Kilmister [16]. The work essentially culminated in his lucid and brilliantly written monograph, *Space-Time Structure* [17]. In it Schrödinger succinctly derives the general form for a non-symmetric connection that is at the same time 'compatible' with the requirement that an interval be defined in the space. Put another way, the general non-symmetric connection satisfies the requirement that the covariant derivative of the metric tensor vanishes under that connection, thereby assuring preservation of length upon parallel transport. We reproduce here Schrödinger's general non-symmetric connection:

$$\Gamma^{\lambda}_{\mu\nu} = -\{{}^{\lambda}_{\mu\nu}\} + g^{\lambda\sigma} g_{\mu\alpha} \Gamma^{\alpha}_{[\nu\sigma]} + g^{\lambda\sigma} g_{\nu\alpha} \Gamma^{\alpha}_{[\mu\sigma]} + \Gamma^{\lambda}_{[\mu\nu]} \tag{32}$$

where the antisymmetric part of the connection is the last term,

$$\Gamma^{\lambda}_{[\mu\nu]} = \tfrac{1}{2}(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}). \tag{33}$$

Schrödinger notes that the general non-symmetric connection is fully determined from the metric tensor and the antisymmetric part alone. In developing equation (32), he first derives the symmetric part consisting of the Christoffel symbol and the next two terms. This is all that he can get from his analysis. He then simply adds the 'arbitrary' antisymmetric part—the last term—to complete the general form. Since he derived the symmetric component explicitly in terms of the antisymmetric part, this last stroke seems quite natural and significant. However, he goes on to say that since this term does not contribute to the geodesic motion then '...let us just scrap it' [17], p 67. He then develops an alternative non-symmetric connection, the 'star affinity', which is still used to this day in non-symmetric gravity theories [18].

We now show that, in fact, Schrödinger lost significant generality by dropping the antisymmetric part of this earlier connection. Let us rewrite our electrodynamic connection and compare it with the Schrödinger connection:

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = -\{{}^{\lambda}_{\mu\nu}\} + \tfrac{1}{2}\kappa g^{\lambda\sigma} g_{\mu\alpha} u^{\alpha} F_{\nu\sigma} + \tfrac{1}{2}\kappa g^{\lambda\sigma} g_{\nu\alpha} u^{\alpha} F_{\mu\sigma} + \tfrac{1}{2}\kappa u^{\lambda} F_{\mu\nu}. \tag{34}$$

We immediately recognize that they are identical in form. Indeed, it is easy to verify that the covariant derivative of this connection vanishes as required by 'compatibility'. Furthermore there is the added benefit that Schrödinger's heretofore unknown antisymmetric components are now identified with the desirable electromagnetic fields consistently in all three terms. It is precisely this last purely antisymmetric term—the one Schrödinger discarded—that gives rise to the Thomas precession in our classical spin equation of motion.

Evidently we can identify the antisymmetric part as

$$\tilde{\Gamma}^{\alpha}_{[\mu\lambda]} \equiv S^{\alpha}_{.\mu\lambda} = \tfrac{1}{2}\kappa u^{\alpha} F_{\mu\lambda} \tag{35}$$

which serves geometrically as the torsion tensor of our space.

6. Torsion in the geometry of electrodynamics

Having discovered this, we are now in a position to demonstrate that our 'electrodynamic connection' can in fact be derived independently very simply by assuming the torsion, equation (35), for a non-Riemannian manifold as described by Schouten [3].

From Schouten's *Ricci calculus*, we find the non-symmetric connection of Schrödinger in its most general form, satisfying his compatibility condition:

$$\tilde{\Gamma}^{\alpha}_{\mu\lambda} = \Gamma^{\alpha}_{\mu\lambda} + S^{..\alpha}_{\mu\lambda} - S^{.\alpha}_{\mu.\lambda} + S^{\alpha}_{.\mu\lambda} \tag{36}$$

where the symmetric $\Gamma^{\alpha}_{\mu\lambda}$ is defined as above being careful to consider our index placement with respect to Schouten's. Then, substituting (35), taking care with indices, we arrive at our own electrodynamic connection:

$$\tilde{\Gamma}^{\alpha}_{\mu\lambda} = \Gamma^{\alpha}_{\mu\lambda} + \tfrac{1}{2}\kappa g^{\alpha\sigma}(u_{\mu}F_{\lambda\sigma} + u_{\lambda}F_{\mu\sigma} - u_{\sigma}F_{\lambda\mu}). \tag{37}$$

This affirms Schrödinger's derivation and the physical significance of the present work as it relates to geometric torsion. Specifically, torsion is directly responsible for at least the Thomas precession.

7. Conclusion

We have derived a new connection from first principles on the simple assumption that the electromagnetic field acts to deform basis vectors of a space undergoing parallel transport. The connection itself consists of gravitational and electromagnetic 'force terms' rather than electromagnetic potentials. This seems desirable and contrasts other Weyl-type approaches.

The connection was shown to generate, not only the Lorentz force for a massive charge but also, for the first time, the classical spin motion with the precise Thomas precession term. This equation provided a counter example that clearly pointed to a loss of generality in Schrödinger's affine theory. Furthermore it should be stressed that the defined torsion tensor alone yields the Lorentz force equation and Thomas precession. Thus it has been shown that torsion can indeed have physical significance.

It would appear that, in light of all this, we have made progress in at least geometrically relating gravitation and electromagnetism. Though this may be the case, a connection alone does not suffice. Rather, we must have metric solutions to field equations. It is at this juncture that we deviate from the rest of the work in Schrödinger's little book. We shall show in a future paper that this connection can be utilized as an operational 'template' with which to embed classical electrodynamics into a non-Riemannian manifold from exact solutions to new Einstein equations. These new equations follow directly from the inclusion of the antisymmetric term in the electrodynamic connection, the torsion—precisely the term Schrödinger rejected.

Appendix

In section 2 we provided the basic definitions required to develop the deformation theory in going from an undeformed to a deformed frame. This was necessary in order to understand what was meant by a deformed connection $\bar{\Gamma}$ as described by equation (13). For the remainder of this derivation it is helpful to refer to the transform diagram in figure A1. The undeformed region includes the points $P\{x\}$ and $Q\{x + \mathrm{d}x\}$ while the deformed region includes $\bar{P}\{\bar{x}\}$ and $\bar{Q}\{\bar{x} + \mathrm{d}\bar{x}\}$. Braces refer to locations. The vectors $\boldsymbol{v}$ and $\bar{\boldsymbol{v}}$ always refer to the points P and $\bar{P}$ respectively while vectors $\boldsymbol{w}$ and $\bar{\boldsymbol{w}}$ refer to points Q and $\bar{Q}$. Recall x and $\bar{x}$ are really the same physical point and have only been distinguished to prevent confusion during the deformation operation.

The program is as follows. Beginning in the deformed state at the point $\bar{Q}\{\bar{x} + \mathrm{d}\bar{x}\}$, we shall parallel transport $\bar{\boldsymbol{w}}$ from $\bar{Q}\{\bar{x} + \mathrm{d}\bar{x}\}$ to $\bar{P}\{\bar{x}\}$ giving $\bar{\boldsymbol{v}}$ at $\bar{P}$. This is followed by an inverse deformation upon $\bar{\boldsymbol{v}}$ from $\bar{P}\{\bar{x}\}$ to $P\{x\}$ leaving us with $\boldsymbol{v}$ at P. On the other hand, we can also arrive at P by first applying an inverse deformation to $\bar{\boldsymbol{w}}$ getting us to $\boldsymbol{w}$ at

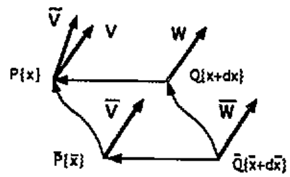

Figure A1. Parallel deformation diagram.

$Q\{x+\mathrm{d}x\}$ followed by parallel transport in the undeformed frame from $Q\{x+\mathrm{d}x\}$ to $P\{x\}$. This gives us the vector $\tilde{v}$ at P not necessarily equal to v. We shall require, however, that $\tilde{v} = v$ at P so that the operations of deformation and parallel transport commute. We must have this to ensure that the length of the vector v is preserved under a parallel deformation. This can only be done at the expense of something else absorbing the effects of this transformation and that something else will be the connection. So it will be the final 'effectively deformed' connection that we shall label $\tilde{\Gamma}\{P\}$ to distinguish it from the initial undeformed (gravitational) $\Gamma\{P\}$. In addition let us recall that $\bar{\Gamma} = \Gamma(G)$ was the source of the inadequate equation (13). We had lost sight of the fact that $\Gamma\{\bar{P}\} \neq \Gamma(G)$ which is to say that the deformed connection is not obtained simply by a substitution of the deformed metric tensor G for the g in the undeformed connection. This will have to be treated separately as well. However, from the above procedure we shall first relate $\tilde{\Gamma}\{P\}$ to $\Gamma\{\bar{P}\}$.

We begin in the deformed state by parallel transporting $\bar{w}$ from $\bar{Q}$ to $\bar{P}$ giving us

$$\bar{v}^\alpha = \bar{w}^\alpha - \Gamma^\alpha_{\beta\gamma}\{\bar{P}\}\bar{w}^\beta\,\mathrm{d}\bar{x}^\gamma \tag{A1}$$

at the point $\bar{P}$. We then apply an inverse deformation to $\bar{v}$ from $\bar{P}$ to P leaving us with

$$v^\alpha = \bar{v}^\alpha - \kappa\bar{v}^\beta\zeta^\alpha_\beta\{P\} \tag{A2}$$

at the point P. Taking the alternate route, we apply a deformation to $\bar{w}$ from $\bar{Q}$ to Q giving

$$w^\alpha = \bar{w}^\alpha - \kappa\bar{w}^\beta\zeta^\alpha_\beta\{Q\} \tag{A3}$$

at Q and then parallel transport w from Q to P leaving finally

$$\tilde{v}^\alpha = w^\alpha - \tilde{\Gamma}^\alpha_{\beta\gamma}\{P\}w^\beta\,\mathrm{d}x^\gamma \tag{A4}$$

at P once again. We have placed a tilde over the connection and hence over v at P in order to distinguish these from the corresponding quantities had we originated in the undeformed state. Also we shall only keep terms to first order in κ and $\mathrm{d}x$. Substituting (A1) into (A2), (A3) into (A4), equating (A2) and (A4) as described earlier and simplifying, we get:

$$\begin{aligned}&-\Gamma^\alpha_{\beta\gamma}\{\bar{P}\}\bar{w}^\beta\,\mathrm{d}\bar{x}^\gamma - \kappa\zeta^\alpha_\beta\{P\}\bar{w}^\beta + \kappa\zeta^\alpha_\sigma\{P\}\Gamma^\sigma_{\beta\gamma}\{\bar{P}\}\bar{w}^\beta\,\mathrm{d}\bar{x}^\gamma \\ &\qquad = -\kappa\bar{w}^\beta\zeta^\alpha_\beta\{Q\} - \bar{w}^\beta\tilde{\Gamma}^\alpha_{\beta\gamma}\{P\}\,\mathrm{d}x^\gamma + \kappa\bar{w}^\beta\zeta^\sigma_\beta\{P\}\tilde{\Gamma}^\alpha_{\sigma\gamma}\{P\}\,\mathrm{d}x^\gamma.\end{aligned} \tag{A5}$$

Now we use

$$\mathrm{d}\bar{x}^\gamma = \mathrm{d}x^\gamma + \kappa\zeta^\gamma_\sigma\{P\}\,\mathrm{d}x^\sigma \tag{A6}$$

for the deformation of the coordinate differential and define

$$\tilde{\Gamma}^{\alpha}_{\beta\gamma}\{P\} = \Gamma^{\alpha}_{\beta\gamma}\{\bar{P}\} + \kappa\delta\Gamma^{\alpha}_{\beta\gamma}. \tag{A7}$$

This definition is a matter of convenience as we seek $\delta\Gamma$ here and we shall shortly relate $\Gamma\{\bar{P}\}$ to our $\Gamma(G)$ and hence to the initial undeformed (gravitational) connection at P, namely $\Gamma\{P\}$. We can also expand the deformation operator $\zeta\{Q\}$ about P giving

$$\zeta^{\alpha}_{\beta}\{Q\} = \zeta^{\alpha}_{\beta}\{P\} + \partial_{\sigma}\zeta^{\alpha}_{\beta}\,\mathrm{d}x^{\sigma}. \tag{A8}$$

Substituting equations (A6)–(A8) in equation (A5) and using the aforementioned approximations yields after some manipulation:

$$\kappa\bar{w}^{\beta}\,\mathrm{d}x^{\sigma}\,[\zeta^{\gamma}_{\sigma}\Gamma^{\alpha}_{\beta\gamma}\{\bar{P}\} - \zeta^{\alpha}_{\tau}\Gamma^{\tau}_{\beta\sigma}\{\bar{P}\} + \zeta^{\tau}_{\beta}\Gamma^{\alpha}_{\tau\sigma}\{\bar{P}\} - \delta\Gamma^{\alpha}_{\beta\sigma} - \partial_{\sigma}\zeta^{\alpha}_{\beta}] = 0. \tag{A9}$$

The bracketed quantity must vanish in general leaving:

$$\delta\Gamma^{\alpha}_{\beta\sigma} = -\partial_{\sigma}\zeta^{\alpha}_{\beta} + \zeta^{\tau}_{\beta}\Gamma^{\alpha}_{\tau\sigma}\{\bar{P}\} - \zeta^{\alpha}_{\tau}\Gamma^{\tau}_{\beta\sigma}\{\bar{P}\} + \zeta^{\gamma}_{\sigma}\Gamma^{\alpha}_{\beta\gamma}\{\bar{P}\}. \tag{A10}$$

Now the first three terms are simply the covariant derivative of ζ^{α}_{β} at $\bar{P}$ leaving:

$$\delta\Gamma^{\alpha}_{\beta\sigma} = -\zeta^{\alpha}_{\beta;\sigma}\{\bar{P}\} + \zeta^{\gamma}_{\sigma}\Gamma^{\alpha}_{\beta\gamma}\{\bar{P}\}. \tag{A11}$$

Thus, from equation (A7), we have for our effectively deformed connection at P:

$$\tilde{\Gamma}^{\alpha}_{\beta\sigma}\{P\} = \Gamma^{\alpha}_{\beta\sigma}\{\bar{P}\} - \kappa\zeta^{\alpha}_{\beta;\sigma}\{\bar{P}\} + \kappa\zeta^{\alpha}_{\beta\gamma}\{\bar{P}\}. \tag{A12}$$

We must now find $\Gamma\{\bar{P}\}$. The second term will provide us with the correction we seek for our effectively deformed connection but for the moment we can do nothing about the third 'extra' term. Indeed Dienes also arrived at precisely such a term in his own transformations and seemed to suggest it was undesirable in some sense. However, using our extended approach, we shall find that this term will exactly cancel out upon evaluation of $\Gamma\{\bar{P}\}$.

To determine $\Gamma\{\bar{P}\}$ we must go back to the definition of the connection, at $\bar{x}$, in terms of the parallel transport of basis vectors, that is:

$$\Gamma^{\alpha}_{\beta\sigma}\{\bar{P}\} = \boldsymbol{E}^{\alpha}\cdot(\partial_{\bar{\sigma}}\boldsymbol{E}_{\beta}). \tag{A13}$$

where

$$\partial_{\bar{\sigma}} = \partial_{\sigma} - \kappa\zeta^{\alpha}_{\sigma}\partial_{\alpha}. \tag{A14}$$

This follows from the vanishing of the covariant derivative of the basis vector E_{β}. On the other hand, we have already obtained equation (13) by simply substituting G for g in the original connection, $\Gamma\{P\}$, giving us $\Gamma(G)$. It is readily seen that what we had in fact done was to simply substitute the deformed $\boldsymbol{E}$ basis for the original $\boldsymbol{e}$ basis so that

$$\Gamma^{\alpha}_{\beta\sigma}(G) = \boldsymbol{E}^{\alpha}\cdot(\partial_{\sigma}\boldsymbol{E}_{\beta}) \tag{A15}$$

and this then is really the $\bar{\Gamma}$ of equation (13).

This can now be easily related to $\Gamma\{\bar{P}\}$ using equations (A13) and (A14):

$$E^{\alpha} \cdot (\partial_{\bar{\sigma}} E_{\beta}) = E^{\alpha} \cdot (\partial_{\sigma} E_{\beta}) - \kappa E^{\alpha} \cdot (\zeta_{\sigma}^{\tau} \partial_{\tau} E_{\beta}) \tag{A16}$$

or

$$\Gamma^{\alpha}_{\beta\sigma}\{\bar{P}\} = \Gamma^{\alpha}_{\beta\sigma}(G) - \kappa \zeta_{\sigma}^{\tau} \Gamma^{\alpha}_{\beta\tau}(G). \tag{A17}$$

Our expression for $\tilde{\Gamma}$, equation (A12), becomes, keeping order κ terms:

$$\tilde{\Gamma}^{\alpha}_{\beta\sigma}\{P\} = \Gamma^{\alpha}_{\beta\sigma}(G) - \kappa \zeta^{\alpha}_{\beta;\sigma} + \kappa \zeta^{\gamma}_{\sigma} \Gamma^{\alpha}_{\beta\gamma}\{\bar{P}\} - \kappa \zeta^{\tau}_{\sigma} \Gamma^{\alpha}_{\beta\tau}(G). \tag{A18}$$

The last two terms cancel to order κ eliminating the 'extra' term and leaving, finally, the desired correction:

$$\tilde{\Gamma}^{\alpha}_{\beta\sigma}\{P\} = \Gamma^{\alpha}_{\beta\sigma}(G) - \kappa \zeta^{\alpha}_{\beta;\sigma}. \tag{A19}$$

Using equations (8) and (9) this last term can be rewritten as

$$\zeta^{\lambda}_{\mu;\nu} = e^{\lambda} \cdot k_{\mu;\nu} \tag{A20}$$

since covariant differentiation passes through the basis, thus providing the exact correction that yields equation (14), the 'effective' deformed connection.

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