INTRINSIC GEOMETRY OF CURVES AND THE MINKOWSKI FORCE

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The Frenet–Serret equations for a curve in a Riemann space are used to derive a theorem regarding the Minkowski Force. The consequence is that the well-known Lorentz–Dirac equation, involving radiation reaction is already implicit in the geometry.

The Lorentz–Dirac equation [1–3] describes the motion of an electron subject to an external electromagnetic force and a self-force due to its own radiation field. The external force on the particle is a Minkowski force given by

$$F^{\mu} = m\dot{v}^{\mu} - \tfrac{2}{3} e^2 [\ddot{v}^{\mu} + v^{\mu}(\dot{v}^{\alpha}\dot{v}_{\alpha})] , \qquad (1)$$

satisfying identically,

$$F^{\mu}v_{\mu} = 0, \quad C \equiv 1 , \qquad (2)$$

where

$$v^{\mu} = dx^{\mu}/ds = (v^0, v^i) = (\gamma, \gamma\beta^i), \quad \mu = 0,1,2,3 ,$$

and $v^{\mu}v_{\mu} = +1$ with metric signature $(+ - - -)$. The terms shown in eq. (1) are the first two "point charge" terms of an infinite series that represents the force on an arbitrary distribution of the electron's charge. Solving this equation for a step input force results in solutions that either blow up in time (runaways) or, upon eliminating these, in a solution wherein the particle accelerates prior to the application of the force. Thus, causality is violated, albeit over a very short "unobservable" time interval of the order of the classical electron radius divided by the speed of light.

Eq. (1) has traditionally been derived using arguments having some physical content. We shall show that the form of eq. (1) is determined exactly by geometrical considerations alone via the general theory of curves in a riemannian manifold.

The Frenet–Serret (FS) equations relate absolute derivatives of the basis vectors on a curve to the same basis vectors and thus form a closed set of relationships for a curve $\mathcal{C}$ possessing four basis vectors (an orthonormal tetrad – or OT) in a riemannian space–time. The basis is normalized in a manner similar to that of Synge [4]. General properties can be found in ref. [5]. We shall label this basis T, N, B, C, respectively, the tangent, normal, binormal and trinormal vectors of the 4-curve where T represents the 4-components T^{μ} ($\mu = 0,1,2,3$), etc. The normalizations are given by:

$$(TT) = T^{\mu}T_{\mu} = +1 = v^{\mu}v_{\mu} ,$$

$$(NN) = -1, \quad (BB) = -1, \quad (CC) = -1 ,$$

with all other inner products vanishing. The FS equations are:

$$\dot{T} = \kappa N, \quad \dot{N} = \kappa T + \tau B, \qquad (3)$$
$$\dot{B} = -\tau N + \eta C, \quad \dot{C} = -\eta B .$$

The dot represents absolute differentiation with respect to s, the arc-length parameter on $\mathcal{C}$ (proper time). κ, τ, η are respectively the first, second and third curvatures of $\mathcal{C}$ and are in general functions of s.

An arbitrary vector F on $\mathcal{C}$ can be resolved along the OT,

$$F = \alpha T + \sigma N + \rho B + \lambda C .$$

We now assume that F is a Minkowski vector, that is, $(FT) = 0$. It follows that α must vanish and we are left with a space-like vector

$$F = \sigma N + \rho B + \lambda C.$$

Our goal will be to represent N, B, C as functions of T and its derivatives via the FS equations. By manipulations and appropriate differentiations of eqs. (3) together with identities derived from three successive differentiations of the invariant $[(TT) = 1]$, we obtain:

$$N = \dot{T}/\kappa ,$$

$$B = -(\dot{\kappa}/\kappa^2\tau)\,\dot{T} + (1/\kappa\tau)\,[\ddot{T} + T(\dot{T}\dot{T})] ,$$

$$C = (1/\kappa\tau\eta)\{[\dddot{T} + 3T(\dot{T}\ddot{T})] - (2\dot{\kappa}\tau + \kappa\dot{\tau})\,B$$
$$- (\ddot{\kappa}/\kappa + \kappa^2 - \tau^2)\,\dot{T}\} .$$

Collecting terms and writing v^μ for T we get the desired result:

$$F^\mu = \alpha_1 \dot{v}^\mu + \alpha_2\,[\ddot{v}^\mu + v^\mu(\dot{v}^\alpha \dot{v}_\alpha)]$$
$$+ \alpha_3\,[\dddot{v}^\mu + 3v^\mu(\dot{v}^\alpha \ddot{v}_\alpha)] . \qquad (4)$$

Thus one can state the *Theorem*: The Minkowski force must be of the form of eq. (4) with no higher-order derivatives in v^μ than the third by virtue of the linear independence of the OT on $\mathcal{C}$ in a 4-space.

The coefficients α_1, α_2, and α_3 are scalar functions of $\kappa, \tau, \eta, \sigma, \rho, \lambda$. The first and second terms can be identified with eq. (1) if α_1 and α_2 are the constants m and $-\frac{2}{3}e^2$ respectively. The third term of (4), a "pseudo-structure" term, is new and does not occur in any previous derivation. Its coefficient cannot be determined by correspondence with a classical expansion since such expansions require some arbitrary charge distribution to determine an infinite series of terms beyond the second of eq. (1). Neither can it be determined from the nonrelativistic quantum version of the Abraham–Lorentz equation derived by Moniz and Sharp [6] since in their work even-order time derivatives of x did not contribute. The purely mathematical result obtained in the present work should be valid for the classical regime and for the quantum mechanical calculation of the motion. The Moniz and Sharp work, as applied here, suggests that the pseudo-structure term might appear in a covariant quantum derivation of the Lorentz–Dirac equation. As shown below, this term can significantly modify solutions of the Lorentz–Dirac equation.

In the non-relativistic limit, retaining the pseudo-structure term, eq. (4) becomes

$$-a\dddot{v} - b\ddot{v} + m\dot{v} = F,$$

where

$$\alpha_1 = m, \quad \alpha_2 = -b, \quad \alpha_3 = -a .$$

For constant $a > 0$ and $F(t)$ the step function given by

$$F(t) = 0, \qquad t \leqslant 0,$$
$$= F = \text{constant}, \qquad t > 0,$$

a completely physical solution is found for the conditions that v be bounded and $v\,(0)$ = constant:

$$\dot{v} = 0, \qquad t \leqslant 0,$$
$$\dot{v} = (F/m)(1 - \mathrm{e}^{-t/\tau}), \qquad t > 0,$$

where

$$\tau = 2a/[b + (b^2 + 4ma)^{1/2}] .$$

The behavior is the usual second-order response to a step input and thus potentially removes the acausality inherent in the Lorentz–Dirac equation in a straightforward way. Indeed, Moniz and Sharp showed that their quantum derivation guaranteed a stable causal solution in the point limit.

It must also be mentioned that Rohrlich [3] has used the FS equations in their 3-dimensional form to help find solutions of the Lorentz–Dirac equation. He stresses the significance of this being more than merely coincidental. The reason for this coincidence is now clear. The Lorentz–Dirac equation is implicitly contained in the FS equations.

In summary, we see that there are not an arbitrary number of vectors that are functions of v and its derivatives satisfying the definition of Minkowski force on a curve in 4-space but rather only three, two of which are present in the Lorentz–Dirac equation.

Furthermore, causality is "built into" relativity. The failure of causality in the apparently covariant Lorentz–Dirac equation suggests that it is not the correct covariant expression – that something is missing. It is, in fact, generally assumed that quantum mechanics must play a role at these small distances. The Moniz and Sharp work confirms this assumption. If the pseudo-structure term cannot be found in classical physics then perhaps quantum theory will provide the correct description from the covariant quantum derivation of the Lorentz–Dirac equation.

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